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Generalized Linear Random Vibration Analysis Using Autocovariance Orthogonal Decomposition

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Application of a stationary Gaussian random process to describe a nondeterministic forcing function of a linear vibrating system is well studied and documented. Two algorithms, Karhunen–Loeve expansion method, and collocation technique for nonstationary and non-Gaussian forcing processes (narrow- or broadband, but not white noise) for linear dynamic systems are developed here. In the Karhunen–Loeve expansion method, the forcing random process autocovariance is decomposed using the well-known Karhunen–Loeve expansion. In the Karhunen–Loeve expansion, Galerkin projection (the weighted-residual method) and collocation technique (discretized covariance matrix) are used to get the eigenvalues and the eigenfunctions/eigenvectors of the autocovariance function, numerically. The steady-state and the transient response of a single-degree-of-freedom system for an exponential autocovariance (Gaussian random process) is obtained using three methods: 1) analytical, 2) semi-analytical and 3) numerical. In the semi-analytical method, the eigenvalues and the eigenfunctions of the exponential autocovariance function are obtained analytically by solving the Fredholm integral equation of the second kind and those definitions of the eigenfunctions and the eigenvalues are used to obtain the numerical response of the single-degree-of-freedom system. In the case of the steady-state response of the single-degree-of-freedom system, the convergence of the standard deviation of the response is shown to be a function of the number of Karhunen–Loeve expansion terms used in the expansion of the autocovariance of the forcing function. The transient analysis of the same single-degree-of-freedom system is carried out using an exponentially modulated nonstationary process. Comparison of the proposed methods with respect to the analytical solutions are presented for both the stationary and the nonstationary Gaussian excitations. The steady-state responses as well as the transient responses for non-Gaussian random processes (uniform, triangular, and beta) for the same single-degree-of-freedom system are also presented.

Nomenclature

C_{FF}	=	excitation covariance matrix obtained by collocation
c	=	damping coefficient, N s/m
d_{jk}	=	h_j basis-function participation factor of ϕ_k eigenvector
$f(t)$	=	band-limited excitation, N
h_j	=	Karhunen–Loeve eigenvector basis function
k	=	stiffness, N/m
m	=	mass, kg
R_{FF}	=	autocovariance of the excitation
R_{XX}	=	autocovariance of the displacement
S_{FF}	=	power spectral density function of the excitation
S_{XX}	=	power spectral density function of the displacement
x	=	displacement, m
ζ	=	nondimensional damping coefficient
λ_c	=	eigenvalues of the excitation covariance matrix
λ_i	=	eigenvalues of the autocovariance function
ξ_i	=	independent identically distributed random variables with zero mean and variance equal to one
$\sigma_x(t)$	=	standard deviation of the displacement
ϕ_c	=	eigenvectors of the excitation covariance matrix
$\phi_i(t)$	=	eigenfunctions of the autocovariance function
ω	=	natural frequency, rad/s

I. Introduction

THE present work is motivated by the design of a fiber-optics pressure and temperature sensor for the use in advanced liquid propulsion systems against fatigue failure. The temperature sensor (no greater than $\frac{1}{4}$ in. diameter) will be housed in a cryogenic piping system to measure the pressure and temperature of different propulsive fuels. Designing such a pressure and temperature sensor is a fluid–structure interaction problem where the sensor vibrations and fluid flows are coupled. Flow-induced vibrations of the temperature sensors are traditionally predicted through the use of the American Society of Mechanical Engineers Boiler and Pressure Vessel (BPV) Code, Sec. III, Appendix N-1300 [1]. BPV code comprises of two main areas of analysis: vortex induced vibration- and turbulence-induced vibration. Currently, only stationary and Gaussian random loadings generated by the turbulence in the flow is assumed in BPV code, which results in a stationary response of the sensor. But intermittent flow during the switch-off and switch-on conditions create a nonstationary random response that may be higher than the stationary response obtained using the BVP code. The loading on the sensor may be nonstationary, which causes an inherently nonstationary response of the sensor. Apart from such fluid–structure interaction problems of interest to us, gust response of aircraft and the vibrations experienced by vehicles on rough roads are other examples of nonstationary vibrations.

In most linear random vibration analyses, the nondeterministic forcing functions are often assumed to be stationary Gaussian random processes in order to obtain the power spectral density (PSD) functions of the response [2,3]. Thus, the standard deviation of the response or the distribution of the standard deviation in the frequency domain can be obtained. In practical problems, most of the random processes describing the excitation are non-Gaussian; meaning that their marginal probability density functions (PDF) may have a positive real domain, which might be finitely bounded. As Gaussian excitations are infinite in their domain, the response of the system would also be infinite, an impossible situation. For finite bounded excitations, the response would also be finite. However, the response of a system to nonstationary and non-Gaussian random forcing

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functions has not been studied in great depth [4]. The response of the dynamic systems under nonstationary excitations has been studied [5–7] for simple systems using an integral method, meaning that integrations involved in the response calculations are performed using analytical or numerical methods. For finite degree-of-freedom (DOF) systems this is possible, but for large structures, it is computationally prohibitive to solve such integrations. Most of the time, the nonstationary random processes would be an *evolutionary* process meaning that the nonstationary process can be split into a reference stationary process and a modulating function. If the excitations are non-Gaussian processes, they would generally be characterized as Markov and Poisson processes and their applications are generally limited to finite DOF models [7–9]. In the authors' opinions, a general method does not exist that can handle linear random vibration problems that are excited by a more general random forcing function than the often-used stationary Gaussian process. The PSD definition makes sense for the problems subjected to stationary Gaussian process. However, the autocovariance definition of the response as well as that of the input functions are applicable to an excitation defined by a nonstationary random process. The autocovariance is the fundamental property of the random processes as opposed to the PSD. In parallel, efforts have also been made to generate rational non-Gaussian random processes for practical applications [10,11].

While using the autocovariance definition to express a given excitation, one has to first discretize the autocovariance function for the response calculation of the system [12,13]. The Karhunen–Loeve (KL) expansion is one of the ways to discretize the autocovariance function. The KL expansion of the Gaussian random process in the random space (probability space) is similar to Fourier decomposition in the deterministic space, as it decomposes a random process into random variables (basis random variables) and a set of deterministic orthogonal functions in real space (domain of the process). The basis random variables of KL expansion [14] of Gaussian random processes are uncorrelated standard normal variables. The basis random variables, when using the KL expansion, should be independent identically distributed (iid) random variables with zero mean and unit variance. Earlier, Poirion et al. [15] presented the KL expansion of non-Gaussian random processes using Monte-Carlo. Later, in 2002, Sakamoto and Ghanem [16] developed the polynomial chaos decomposition for the non-Gaussian nonstationary processes. Recently, Mulani et al. [17] and Phoon et al. [18] presented the KL expansion of non-Gaussian random processes.

Here, an algorithm is presented that uses the KL expansion for decomposition of the autocovariance into orthogonal components. During the KL expansion of the autocovariance of the forcing function, the time duration for analysis is chosen such that it is greater than the nonzero duration of the autocovariance. Orthogonality of the basis random variables of the KL expansion is used to obtain the standard deviation and the autocovariance of the response. This algorithm is applied to a single-degree-of-freedom (SDOF) system that is excited by a random process having the mean equal to 0 and the variance equal to 1. This random process is described by the stationary exponential and the exponentially modulated nonstationary autocovariance functions. The KL expansion of the exponential autocovariance is carried out using analytical functions, Galerkin projection method [19], and the collocation technique [13].

The Galerkin projection method often yields negative eigenvalues of the autocovariance function and results in numerical inaccuracies during integration while solving the Fredholm equation of the second kind. So a user has to make sure that the eigenvalues of the autocovariance are both positive and finite. One can obtain eigenvalues and eigenvectors of the autocovariance function using the collocation technique [13]. Generally, such an approach does not yield negative and infinite eigenvalues. For the exponential autocovariance function, the PSD definition is available, so the response PSD and its standard deviation are obtained. The standard deviation of the steady-state (stationary excitation) response of SDOF system having the Gaussian excitation is compared with the response obtained using Fourier transform (analytical technique) for two approaches: 1) semi-analytical method, and 2) numerical methods (KL expansion method and collocation technique). In the semi-analytical method, eigenvalues and eigenfunctions of the autocovariance of the excitation are obtained by solving the Fredholm equation of the second kind analytically and definitions of the obtained eigenfunctions are used to construct the response of the system. The autocovariances of the response are compared for semi-analytical and numerical methods. A transient analysis response of the SDOF system for the exponential autocovariance (stationary random process) and the exponentially modulated nonstationary random process are presented. Additionally, the PDFs of the displacement for a non-Gaussian stationary excitation (exponential autocovariance) as well as for a non-Gaussian nonstationary (exponentially modulated) process are obtained, where the excitations are uniform, triangular and beta random processes.

When the KL expansion method and the collocation technique are applied to a SDOF system subjected to stationary and nonstationary random processes as the forcing functions, the obtained standard deviation of the displacement matches very well with the response obtained using analytical methods. The KL expansion method more accurately describes the statistics of the response as compared with the collocation technique but many times suffers from ill-conditioned matrices obtained during the KL expansion of the excitation autocovariance. During the transient analysis of the system, the PDF of the displacement changes from the Dirac-delta function to a PDF having some finite standard deviation, the shape of the PDF of the displacement depends upon the number of terms used for the expansion of the excitation autocovariance during the KL expansion method or the collocation technique. For wideband excitation, the PDF of the displacement is almost Gaussian for both methods. But for short-band excitation or nonstationary response, the PDF of the displacement obtained using the collocation technique may differ with the one obtained using the KL expansion method. Still, the standard deviation obtained using collocation technique matches very well with the obtained standard deviation using the KL expansion method.

II. KL Expansion of the Excitation and the Response Calculation

The proposed algorithms can be easily applied to general systems (multiple degrees of freedom) using commercial finite element method (FEM) software. For derivation convenience, to aid in understanding and from a verification perspective, the proposed algorithms are applied to a SDOF system. A time-invariant linear second-order system subjected to a forcing function can be written as

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{1}$$

where m, c, and k are mass, damping coefficient, and stiffness properties of the system. The forcing function $f(t)$ can range from a narrowband to a broadband (but not white noise) random process and can be a stationary or a nonstationary process. This process may be Gaussian or non-Gaussian and is prescribed by its autocovariance and the mean value. Using the KL expansion, the random excitation $f(t)$ is written as

$$f(t) = \mu_F(t) + \sum_{i=1}^{n} \sqrt{\lambda_i}\, \xi_i \phi_i(t) \tag{2}$$

where $\mu_F(t)$ is the deterministic part of the forcing function. λ_i and ϕ_i are eigenvalues and eigenfunctions of the autocovariance function, and ξ_i are random variables having a mean value equal to zero and a variance equal to 1. Variables λ_i and $\phi_i(t)$ are obtained by solving the following Fredholm equation of the second kind [19]:

$$\int_D R_{FF}(t_1, t_2)\phi_n(t_1)\,\mathrm{d}t_1 = \lambda_n \phi_n(t_2) \tag{3}$$

$$D \equiv [T_{\min}, T_{\max}] \tag{4}$$

$$L = (T_{\max} - T_{\min})/2 \tag{5}$$

where $R_{FF}(t_1, t_2)$ is the autocovariance function of the force/excitation, D is the time domain of the forcing function, $T_{\min}$ and $T_{\max}$ are the lower and upper bounds of the time duration in which the response of the system is sought, and L represents half of the duration of the time domain t. The KL expansion decomposes a random process into deterministic orthogonal components and the iid random variables. For a Gaussian process, these will be standard normal variables, and for non-Gaussian processes, these random variables should be calculated [17] using the PDF definition of the force. Equation (3) can be solved analytically for some special autocovariances and for general covariances, it should be solved using a Galerkin procedure [19].

Let the eigenfunctions of the autocovariance be represented as

$$\phi_k(t) = \sum_{j=1}^{m} d_{jk} h_j(t) \tag{6}$$

where d_{jk} are constants and h_j are user-defined basis functions.

Therefore, for such a time domain, the components of h_j should be chosen such that they form a complete basis of the system, but it is not a necessary condition. The d_{jk} can be viewed as the h_j basis-function participation factors of the eigenfunction, ϕ_k of the excitation autocovariance. For the steady-state response of the dynamic system, h are chosen such that they represent complete cosine and sine functions over the selected time domain and are given as

$$h = \left\{1, \cos\frac{\pi}{L}t, \sin\frac{\pi}{L}t, \ldots, \cos\frac{\pi(n-1)}{L}t, \sin\frac{\pi(n-1)}{L}t\right\} \tag{7}$$

Here, h_j are represented in each position of the complete vector h. For the transient analysis of the dynamic systems, each h_j is chosen such that the basis-function vector h includes multiples of both half-cosine and half-sine functions to represent the response components corresponding to basis functions with lower frequencies, as the system response starts with a frequency of zero. It is important to choose such functions to represent transient response with refined basis functions as compared with basis functions given in Eq. (7), and therefore the vector of basis functions h is given as

$$h = \left\{1, \cos\frac{\pi}{2L}t, \sin\frac{\pi}{2L}t, \ldots, \cos\frac{\pi(n-1)}{2L}t, \sin\frac{\pi(n-1)}{2L}t\right\} \tag{8}$$

Using the definition of the KL expansion of the forcing function as given in Eq. (2), Eq. (1) is written as

$$m\ddot{x} + c\dot{x} + kx = \mu_F(t) + \sum_{i=1}^{n} \xi_i \sqrt{\lambda_i}\left(\sum_{j=1}^{n} d_{ji} h_j(t)\right) \tag{9}$$

As we are interested in both the steady-state and the transient statistics of the SDOF system, the SDOF response that has both the steady-state response and the transient response for the step input, $F = F_0$ with zero initial displacement and velocity, is given as

$$x = \frac{F_0}{k}\left(1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \sin(\omega_d t)\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right) \tag{10}$$

and for the sinusoidal forcing functions $F = F_0 \cos\Omega t$, the total response x_c (which includes both the steady-state and the transient response) is given as [4]

$$x_c = x_{st} + x_t \tag{11}$$

$$x_c = \frac{F_0 \cos(\Omega t + \psi)}{k\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}} + H_c e^{-\zeta\omega_n t}\cos(\omega_d t + \psi_c) \tag{12}$$

where

$$\beta = \frac{\Omega}{\omega_n} \tag{13}$$

$$\psi = \tan^{-1}\left(\frac{-2\zeta\beta}{1-\beta^2}\right) \tag{14}$$

x_t is the transient response, x_{st} is the steady-state response, ζ is the nondimensional damping coefficient, β is the normalized frequency, ω_n is the natural frequency of the SDOF system under consideration, and ψ is the phase angle between the forcing function and the displacement; H_c and ψ_c are obtained by satisfying initial conditions. For zero initial displacement and velocity,

$$H_c = -x_{st}\sqrt{\cos(\psi)^2 + \frac{(\zeta\cos(\psi) - \beta\sin(\psi))^2}{1-\zeta^2}} \tag{15}$$

$$\psi_c = -\cos^{-1}\left(-\frac{x_{st}\cos(\psi)}{H_c}\right) \tag{16}$$

If the forcing function is $F = F_0 \sin\Omega t$, the total response of the SDOF is given as

$$x_s = x_{st} + x_t \tag{17}$$

$$x_s = \frac{F_0 \sin(\Omega t + \psi)}{k\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}} + H_s e^{-\zeta\omega_n t}\sin(\omega_d t + \psi_s) \tag{18}$$

For zero initial displacement and velocity, H_s and ψ_s are given as

$$H_s = -x_{st}\sqrt{\sin(\psi)^2 + \frac{(\zeta\sin(\psi) + \beta\cos(\psi))^2}{1-\zeta^2}} \tag{19}$$

$$\psi_s = -\sin^{-1}\left(-\frac{x_{st}\sin(\psi)}{H_c}\right) \tag{20}$$

For convenience during the derivation, it is assumed that the mean value of the forcing function, $\mu_F(t)$ is zero, the initial displacement and the velocity are deterministic, and assumed to be zero. Even if the initial displacement and the velocity are nonzero, they can be associated with the mean component of the forcing function; if the mean component of the forcing function is zero, then they can be associated with the major component of the KL expansion. If the mean value of the forcing function, $\mu_F(t)$ is a nonzero function of time, then the mean of the nonstationary response will be nonzero and $\mu_F(t)$ will not affect the calculation of the standard deviation of the response. The displacement response of the SDOF is given as

$$x(t) = \sum_{i=1}^{n} \xi_i \sqrt{\lambda_i}\left(\sum_{j=1}^{n} d_{ji} m_j(t)\right) \tag{21}$$

The response functions m_j corresponding to the basis functions h_j of the excitation are obtained using Eqs. (10), (11), and (17) by substituting $F_0 = 1$. While evaluating these responses, β_i and ψ_i are calculated using Eqs. (13) and (14), respectively, and Ω_i for the steady-state and the transient analyses are defined by Eqs. (22) and (23), respectively.

$$\Omega = \left\{0, \frac{\pi}{L}, \frac{\pi}{L}, \ldots, \frac{(n-1)\pi}{L}, \frac{(n-1)\pi}{L}\right\} \tag{22}$$

$$\Omega = \left\{0, \frac{\pi}{2L}, \frac{\pi}{2L}, \ldots, \frac{(n-1)\pi}{2L}, \frac{(n-1)\pi}{2L}\right\} \tag{23}$$

By comparing Eqs. (9) and (21), it is seen that for complex multi-degree-of-freedom structures, the statistics of the response when excited by the stochastic forcing function can be easily obtained. The user can choose the basis functions h_j given by either Eq. (7) or Eq. (8). Then the user has to obtain the eigenvalues λ_i and d_{ji}, the h_j basis-function participation factors of the eigenfunctions, and ϕ_i of the excitation autocovariance outside of commercial FEM software. The response functions m_j corresponding to the basis functions h_j of the excitation should be obtained to construct the PDF or statistics of the response of the system.

Using the orthogonality of iid basis random variables ξ, which are given as

$$\langle \xi_i, \xi_j \rangle = \delta_{ij} \tag{24}$$

where $\langle\,\rangle$ is the mathematical expectation operator, and the covariance $\sigma_x^2(t)$ and autocovariance $R_{XX}(t_1, t_2)$ of the response are given as [19]

$$\sigma_x^2(t) = \sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{n} d_{ji} m_j(t) \right)^2 \tag{25}$$

$$R_{XX}(t_1, t_2) = \langle x(t_1), x(t_2) \rangle \tag{26}$$

$$= \sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{n} d_{ji} m_j(t_1) \right) \left(\sum_{k=1}^{n} d_{ki} m_k(t_2) \right) \tag{27}$$

From Eq. (25), it can be seen that the standard deviation of the response depends upon the eigenvalues of the forcing autocovariance function, participation factors of the chosen basis functions, and the deterministic system response of the basis functions. Equation (27) indicates that the autocovariance of the response depends upon the deterministic responses at two different times of the chosen basis functions.

III. Autocovariance Decomposition Using Collocation Method and the Response Calculation

At times, a number of eigenvalues of a positive semi-definite autocovariance function become negative or infinite when obtained using the KL expansion and the Galerkin procedure because of the basis functions used. Similarly, sometimes the chosen basis functions may not be optimal in terms of calculations meaning that a minimum number of basis functions cannot represent the given autocovariance [12] as accurately as compared with some other set of basis functions. To avoid negative and infinite eigenvalues of the autocovariance function, the eigenvalues and the eigenvectors of the given autocovariance are obtained from the covariance matrix, by collocating the autocovariance function at discrete points. Generally collocation is used in the finite element method to replace continuous shape functions by Dirac-delta functions, thus using the properties of the Dirac-delta functions [20]; the integrations are evaluated at the desired points only. Here, collocation is used to obtain the eigenvectors of the autocovariance function by converting the autocovariance function into an autocovariance matrix. The collocation technique should be applied to the given autocovariance function in such a way that it results into positive-semidefinite covariance matrix. By forcing a sufficient number of uniform distribution of collocation points, one can obtain positive-semidefinite covariance matrix for almost all autocovariance functions. These collocation points should not be too few; otherwise, one would lose important information provided by the autocovariance function. Also, if these points are chosen to be too close to each other, the collocation technique would result in an ill-conditioned covariance matrix.

Let C_{FF} represent the excitation covariance matrix obtained by collocation of R_{FF}, the excitation autocovariance function, and the elements of C_{FF} matrix are given by Eq. (28):

$$C_{FF}(i, j) = R_{FF}(t_i, t_j) \tag{28}$$

Eigenvalues λ_c and eigenvectors ϕ_c of C_{FF} are obtained using currently available deterministic algorithms in commercial software such as MATLAB and LAPACK, and all λ_c are confirmed to be positive and finite. To obtain the dynamic response of the system using the algorithm explained in Sec. II and the collocation technique, d_{jk} coefficients are obtained by choosing the basis functions given in Eq. (8) and using Eq. (29). The chosen basis functions are orthogonal to each other and satisfy Eq. (30):

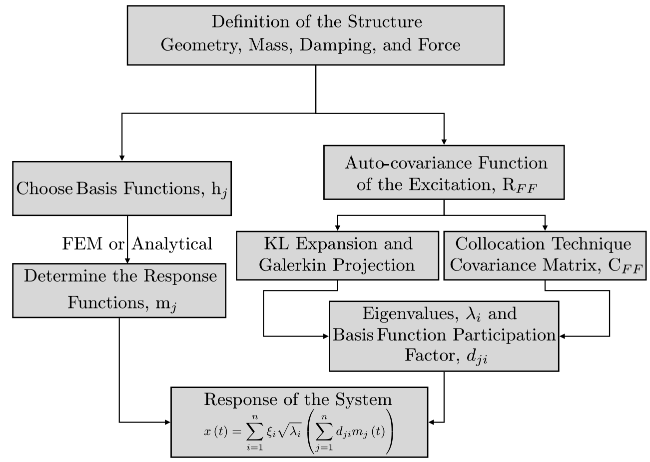


Fig. 1 Flowchart of the KL expansion method and the collocation technique.

$$d_{jk} = \frac{\int_{T_{\min}}^{T_{\max}} \phi_{c_k} h_j \, \mathrm{d}t}{\int_{T_{\min}}^{T_{\max}} h_j h_j \, \mathrm{d}t} \tag{29}$$

$$\int_{T_{\min}}^{T_{\max}} h_i h_j \, \mathrm{d}t = 0 \quad \text{if } i \neq j \tag{30}$$

While obtaining d_{jk}, the basis functions h_i are discretized with the same collocation points that were used to obtain the covariance matrix given in Eq. (28). In Eq. (29), the cross product of discretized vectors such as ϕ_{c_k}, h_i, and h_j is carried out before evaluating various integrals. Once d_{jk} are obtained, the procedure described in Sec. II is followed to obtain the statistics of the response using Eqs. (9–27). The flowchart of the proposed algorithms, the KL expansion method, and the collocation technique are shown in Fig. 1.

IV. Numerical Examples

To check the accuracy of the proposed algorithms, these algorithms are applied to the SDOF system that is excited by the stationary exponential autocovariance random forcing function. This SDOF system is shown in Fig. 2. In this SDOF system, the mass m of the system is 1 kg, stiffness k is 300 N/m. and damping coefficient $\zeta = 0.2$ is chosen such that the response of the system converges to the steady-state response within $t = 1$ s. The steady-state response of the SDOF system is obtained using four different approaches. The convergence study of the standard deviation of the response with respect to number of terms KL expansion is done. The steady-state response to a non-Gaussian stationary exponential excitation is carried out. The transient analysis of the same SDOF system is obtained using numerical techniques (the KL expansion method and the collocation technique) and compared with analytical solution. The SDOF's transient response to a nonstationary exponentially modulated excitation is compared with the solution obtained by numerical integration of the integral generated using convolution theorem and unit impulse response. In addition to this, a non-Gaussian response to an exponentially modulated excitation is also obtained.

A. Exponential Autocovariance

The steady-state and the transient responses of the SDOF are obtained for an exponential autocovariance, which is given in

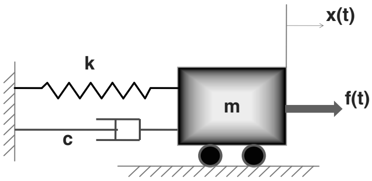


Fig. 2 Single-degree-of-freedom system.

Eq. (31). The applied excitation is a stationary process, its steady state would be stationary but the transient response would be nonstationary:

$$R_{FF}(\tau) = e^{-20|\tau|} \tag{31}$$

1. Analytical Method

For the particular autocovariance defined in Eq. (31), the PSD function given in Eq. (32) can be found by taking the Fourier transform of the autocovariance defined in Eq. (31):

$$S_{FF}(\omega) = \frac{6.366197}{400 + \omega^2} \tag{32}$$

The PSD function of the displacement is shown in Fig. 3 and is given as

$$S_{XX}(\omega) = |H(\omega)|^2 S_{FF}(\omega) \tag{33}$$

where $H(\omega)$ is the frequency response function and is given as

$$|H(\omega)|^2 = \frac{1}{k^2((1-\beta^2)^2 + (2\zeta\beta)^2)} \tag{34}$$

Covariance of the displacement is calculated by integrating the PSD of the response as given in Eq. (35) and the standard deviation of the displacement, σ_X for this particular SDOF system and excitation, is found to be 3.9307 mm:

$$\sigma_X^2 = \int_{-\infty}^{\infty} S_{XX}(\omega)\,\mathrm{d}\omega \tag{35}$$

2. Semi-Analytical Method

The exponential autocovariance described by Eq. (31) is a particular type of the autocovariance whose eigenfunctions and eigenvalues can be obtained using analytical methods. The obtained eigenfunctions of this autocovariance are defined by analytical functions, which would be accurate as compared with the obtained by the weighted-residual method while solving the Fredholm Eq. (3) of the second kind. By differentiating Eq. (3) twice with respect to time and applying appropriate boundary conditions at $t = T_{\min}$ and $t = T_{\max}$, transcendental equations (36) and (39) are obtained. Solutions of Eq. (36) along with Eqs. (37) and (38) define symmetric eigenvalues and eigenfunctions. In the same way, solutions of Eq. (39) along with Eqs. (40) and (41) define antisymmetric eigenvalues and eigenfunctions. Additional information on analytical method used to obtain eigenfunctions and eigenvalues is given [19].

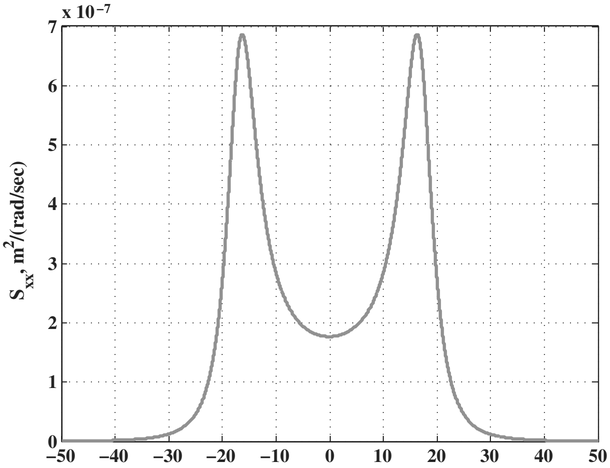


Fig. 3 Power spectral density function of the displacement when excited by exponential autocovariance.

For symmetric eigenfunctions of an exponential autocovariance,

$$C - \kappa \tan(\kappa a) = 0 \tag{36}$$

$$\phi_i(t) = \frac{\cos(\kappa_i t)}{\sqrt{a + \frac{\sin(2\kappa_i a)}{2\kappa_i}}} \tag{37}$$

$$\lambda_i = \frac{2C}{\kappa_i^2 + C^2} \tag{38}$$

For antisymmetric eigenfunctions of an exponential autocovariance,

$$\kappa^* + C\tan(\kappa^* a) = 0 \tag{39}$$

$$\phi_i^*(t) = \frac{\sin(\kappa_i^* t)}{\sqrt{a - \frac{\sin(2\kappa_i^* a)}{2\kappa_i^*}}} \tag{40}$$

$$\lambda_i^* = \frac{2C}{\kappa_i^{*^2} + C^2} \tag{41}$$

where C is the correlation length parameter of the forcing autocovariance, a is equal to the half of the time domain considered in the KL expansion, and κ_i and κ_i^* are the solutions of the transcendental equations (36) and (39). The ϕ_i and ϕ_i^* are symmetric and antisymmetric eigenfunctions of the exponential autocovariance, respectively. The λ_i and λ_i^* are the eigenvalues of the same autocovariance function. Hence, the forcing function and the displacement can be written as summation of orthogonal components obtained using the KL expansion and are given in Eqs. (42) and (43), respectively:

$$F(t) = \sum_{i=1}^{n} \xi_i \sqrt{\lambda_i}\phi_i(t) \tag{42}$$

$$x(t) = \sum_{i=1}^{n} \xi_i \sqrt{\lambda_i} S_i(t) \tag{43}$$

where ϕ_i and S_i are defined in Eqs. (44–47), and σ_i and ν_i are defined in the same way as defined in Eqs. (13) and (14), respectively:

$$\phi_{1,3,5,\ldots,n} = c_i \cos(\eta_i t) \tag{44}$$

$$\phi_{2,4,6,\ldots,n} = e_i \sin(\eta_i t) \tag{45}$$

$$S_{1,3,5,\ldots,n} = \frac{c_i \cos(\eta_i t + \nu_i)}{k\sqrt{(1-\sigma_i^2)^2 + (2\zeta\sigma_i)^2}} \tag{46}$$

$$S_{2,4,6,\ldots,n} = \frac{e_i \sin(\eta_i t + \nu_i)}{k\sqrt{(1-\sigma_i^2)^2 + (2\zeta\sigma_i)^2}} \tag{47}$$

Therefore, the covariance $\sigma_x^2(t)$ and the autocovariance $R_{XX}(t_1, t_2)$ of the response are given as

$$\sigma_X^2(t) = \sum_{i=1}^{n} \lambda_i S_i^2(t) \tag{48}$$

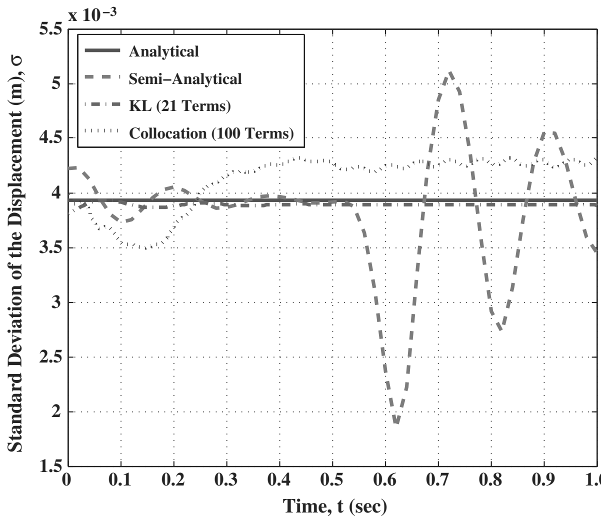


Fig. 4 Exponential autocovariance steady-state standard deviation of the displacement.

$$R_{XX}(t_1, t_2) = \sum_{i=1}^{n} \lambda_i S_i(t_1) S_i(t_2) \tag{49}$$

As the KL expansion is used to decompose autocovariance of the forcing random process, the standard deviation of the displacement shows the Gibbs phenomenon and it depends upon the number of terms used in the KL expansion. The variation of the standard deviation with respect to time is shown in Fig. 4 for the various methods presented here. While calculating the response, 32 eigenfunctions are used to represent input autocovariance. To remove the effects of the Gibbs phenomenon for the stationary excitation, smoothing of the standard deviation is carried out by averaging over the time in which the analysis is carried out, and the mean of the standard deviation comes out to be 3.8228 mm. The semi-analytical method results do not match well with the analytical results at some times, due to numerical inaccuracies during conversion and obtaining responses; however, the purpose of this work is not to explore the intricacies of semi-analytical method.

3. *Numerical Methods*

For general application of these algorithms to nonstationary and non-Gaussian random processes, the KL expansion method and the collocation technique should be used as described in Secs. II and III, respectively. These algorithms are applied to the SDOF system, described earlier. The variation of the standard deviation is shown in Fig. 4. For the steady-state solution, this figure illustrates the advantage of the KL method over the semi-analytical, which appears to become unstable beyond $t = 0.5$ s and the collocation technique with 100 terms. The mean of the standard deviation is 3.88995 mm using the KL expansion method and 4.12461 mm using the collocation technique. The convergence of the standard deviation of the displacement with respect to the number of terms in the KL expansion is shown in Fig. 5. By including more than 17 terms in the KL expansion, the converged steady-state standard deviation of the displacement is obtained when the SDOF system is excited by the exponential autocovariance, defined by Eq. (31). The KL solution offers excellent correlation with the analytical solution for all time and the mean of the standard deviation of 3.88995 mm compares favorably with the value of 3.9307 mm obtained from the analytic method. The KL expansion method results match the analytic results with less than 2% error. The collocation technique results match the analytical method results with less than 10% error. The autocovariances of the displacements using the semi-analytical and the KL expansion methods are shown in Figs. 6a and 6b, respectively. The autocovariance of the displacements is shown in Fig. 7 when obtained using the collocation technique. During application of the KL expansion method, 21 terms are used to represent the eigenvectors and 100 terms are used during the application of the collocation technique.

While describing a stochastic excitation, one needs to define the autocovariance of the random process and the marginal PDF of the process. From the marginal PDF definitions, the iid basis random variables should be defined to obtain the KL expansion of the given random process [17]. Using the same approach [17], iid basis random

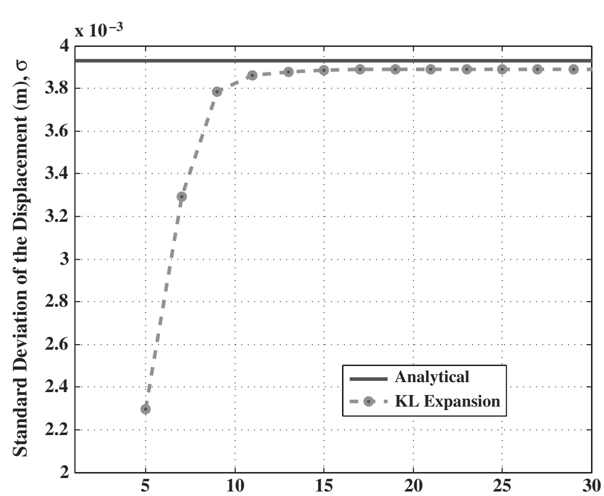
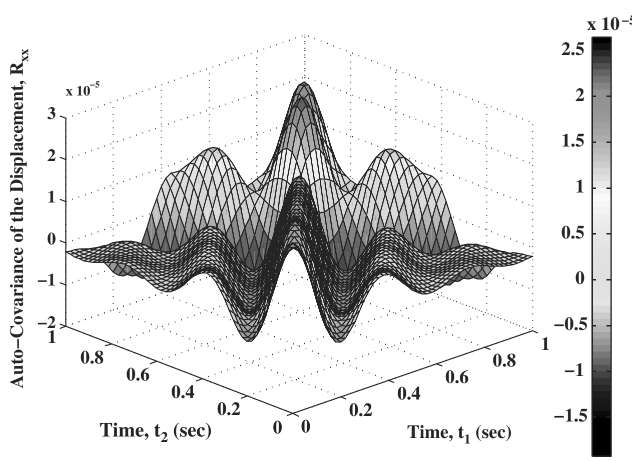
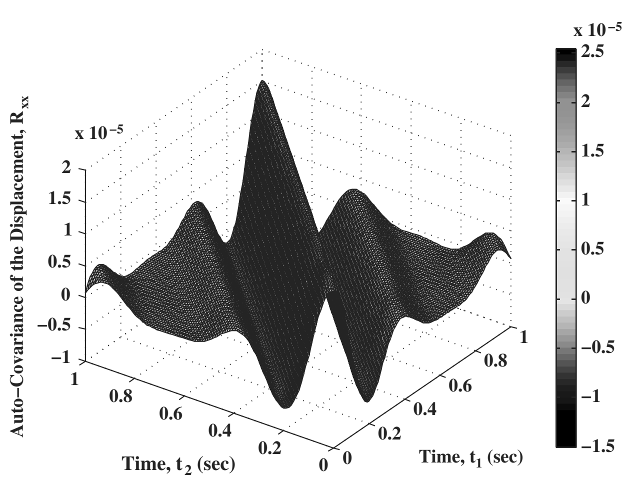


Fig. 5 Convergence of the steady-state standard deviation of the displacement when excited by exponential autocovariance.



a) Semi-analytical method



b) KL expansion method

Fig. 6 Autocovariance of the displacement for $T = [0 \quad 1]$ s when excited by exponential autocovariance.

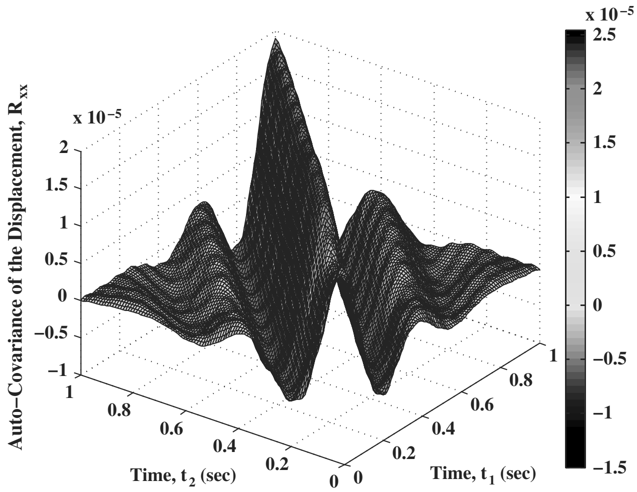


Fig. 7 Autocovariance of the displacement using collocation technique for $T = [0 \quad 1]$ s when excited by exponential autocovariance.

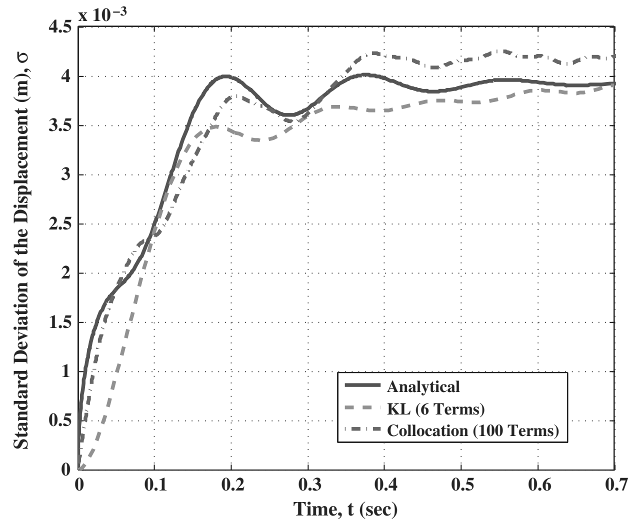


Fig. 9 Exponential autocovariance transient response standard deviation of the displacement.

variables for non-Gaussian random process having zero mean and unit variance are obtained and are shown in Fig. 8a for uniform, triangular, and beta PDFs along with the input Gaussian marginal PDF. Here, it is assumed that the user has obtained the correct marginal PDF. The purpose of using different marginal PDFs for non-Gaussian random process is to show the applicability of the proposed algorithms. The steady-state displacement PDFs are obtained using Eq. (21) and the KL expansion method for these random processes and are shown in Fig. 8b. For all the excitation processes, the steady-state displacement PDFs are almost Gaussian because the displacement calculation requires 21 KL expansion terms and 100 collocation terms.

Lindeberg–Levy–Feller central limit theorem [21] states that as the number of the independent nonidentically distributed variables having finite variance increases in the summation, the linear combination of these variables tends to be a Gaussian. The same can be inferred using Lyapunov condition of the central limit theorem [21], as the number of terms are increasing in the summation of nonidentically independent distributed variables.

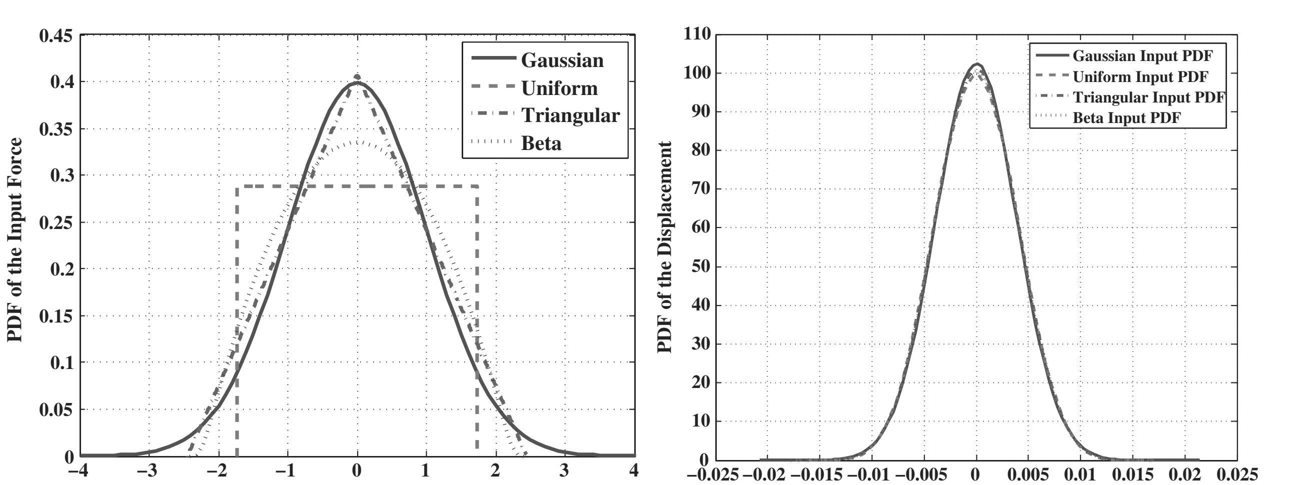
The transient analysis of the same SDOF system having zero initial displacement and velocity and excited by an exponential autocovariance is carried out using both the KL expansion method and the collocation technique. The same definition of the autocovariance is used as defined in Eq. (31). Even though the autocovariance is stationary, the transient response is nonstationary. The transient response obtained using numerical techniques is compared with the analytical solution in Fig. 9. The derivation of the analytical solution is described in the Appendix. The obtained displacement autocovariance using the KL expansion method and the collocation technique are shown in Figs. 10 and 11, respectively. Both the autocovariances match approximately. The transient response is obtained using six KL expansion terms and 100 collocation terms.

B. Exponentially Modulated Autocovariance

Mostly, a nonstationary excitation occurs during initial time. It is important to determine the transient response when excited by the nonstationary excitation because the standard deviation of the displacement during the transient period may be more than the steady-state response and many times, the excitation becomes deterministic after some time of the application of the force. The standard deviation during the transient period changes from 0 to some value and then settles down to a constant value. To check the accuracy of developed algorithms, the same SDOF system is excited by nonstationary excitation as given by Eq. (50):

$$R_{FF}(t_1, t_2) = e^{-(t_1+t_2)} e^{-20|t_1-t_2|} \tag{50}$$

The standard deviation of the displacement for exponentially modulated autocovariance is shown in Fig. 12 when obtained using both the KL expansion method and the collocation technique. In Fig. 12, the analytical solution is also plotted, which is obtained by



a) Input random process marginal PDFs **b) Steady-state displacement PDFs**

Fig. 8 Exponential autocovariance random process marginal PDFs and steady-state displacement PDFs.

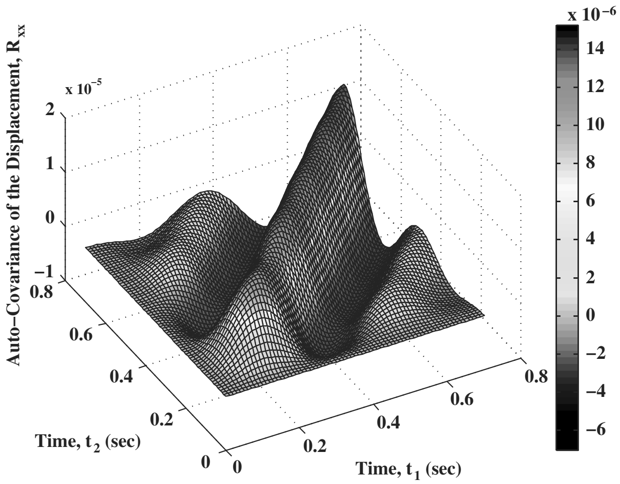


Fig. 10 Transient autocovariance of the displacement for $T = [0 \quad 0.7]$ s for exponential autocovariance using the KL expansion method.

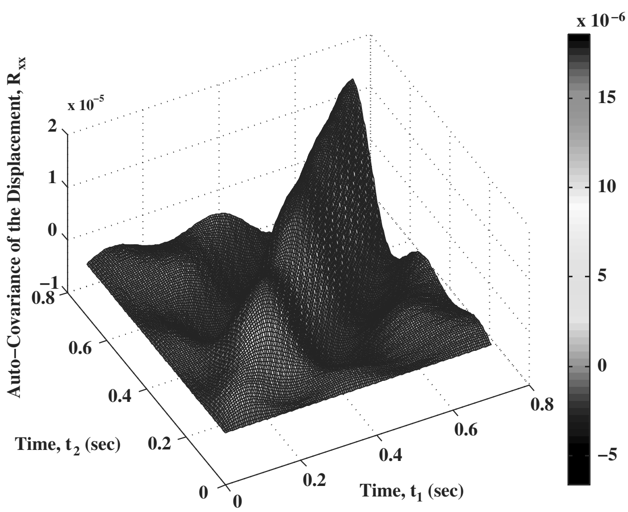


Fig. 11 Transient autocovariance of the displacement for $T = [0 \quad 0.7]$ s for exponential autocovariance using the collocation technique.

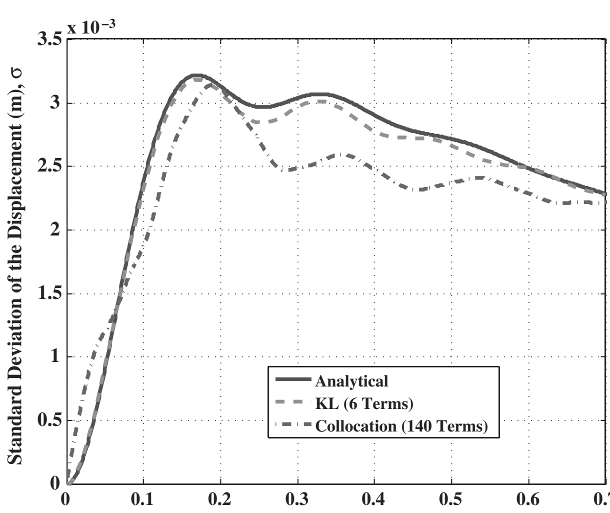
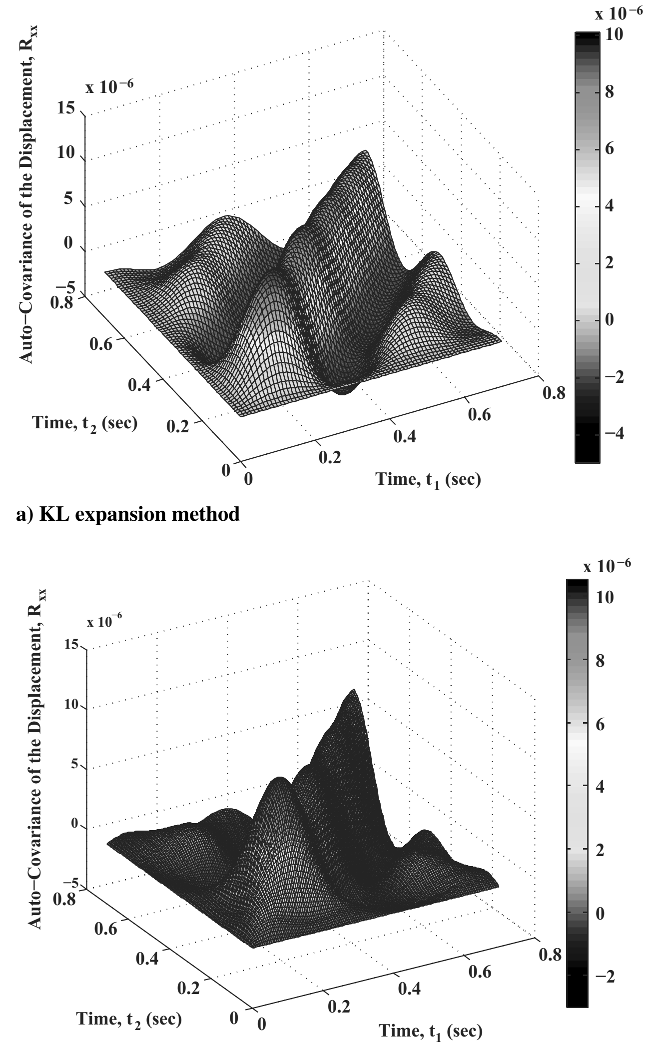


Fig. 12 Exponentially modulated autocovariance transient standard deviation of the displacement.



a) KL expansion method

b) Collocation technique

Fig. 13 Transient autocovariance of the displacement for $T = [0 \quad 0.7]$ s for exponentially modulated autocovariance.

direct integration of Eq. (51). The transient displacement autocovariance is plotted in Fig. 13 for both methods and they match approximately. Equation (51) is obtained by taking the expectation of the product of the responses obtained using the convolution theorem and the unit impulse response [5]:

$$\sigma_x^2(t) = \int_0^t \int_0^s \frac{R_{FF}(r,s) e^{-(\zeta\omega_n(2t-r-s))} \sin(\omega_d(t-r)) \sin(\omega_d(t-s))}{\omega_d^2} \, \mathrm{d}r \, \mathrm{d}s \tag{51}$$

The non-Gaussian response is obtained using the same approach [17], iid basis random variables for non-Gaussian random process having zero mean and unit variance are obtained and are shown in Fig. 8a for uniform, triangular, and beta PDFs along with the input Gaussian marginal PDF. The transient displacement PDFs at $t = 0.17$ s are obtained using Eq. (21) and the KL expansion method for these random processes and are shown in Fig. 14. For all the excitation processes, transient displacement PDFs are non-Gaussian because the displacement calculation requires only six KL expansion terms.

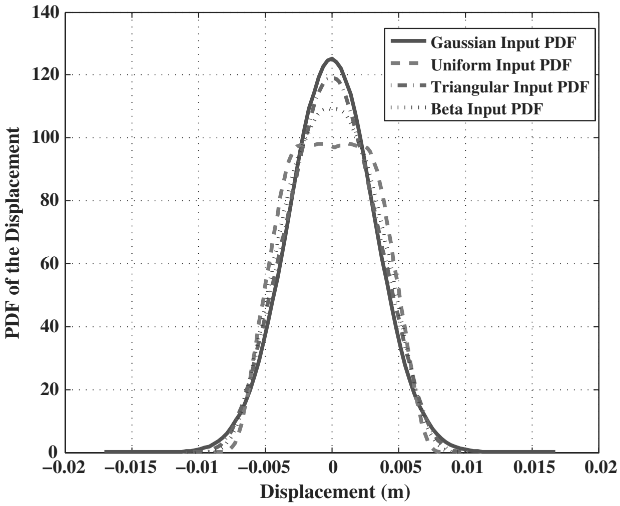


Fig. 14 Exponentially modulated autocovariance transient standard deviation of the displacement.

V. Conclusions

The KL expansion method and the collocation technique are proposed to calculate the response of the dynamic system (second-order) subjected to stationary or nonstationary random processes that might be Gaussian or non-Gaussian. These random processes must be described by the autocovariance functions and generally those are available. The excitation random processes are discretized into orthogonal components and those are used to obtain the statistics of the response. If the autocovariance function correlation length is small, meaning that input random process is wideband, then expansion of the random process has many KL expansion or collocation terms. During the transient analysis, the PDF of the displacement evolves from zero standard deviation to some constant standard deviation. During the evolution of the PDF of the displacement, the standard deviation of the displacement is often be greater than the stationary standard deviation. Even during the evolution of the PDF, the PDF changes its density function, it starts with Dirac-delta at time $t = 0$ s and may become, or tend to become, Gaussian if excitation is wideband (number of terms required during the autocovariance expansion). The KL expansion method more accurately describes the statistics of the dynamic response as compared with the collocation technique but often suffers from ill-conditioned matrices that are obtained during the KL expansion of the autocovariance function. The collocation technique generally overpredicts the system response as compared with the KL expansion method. During the transient analysis, the standard deviation predicted by both the methods are same but the PDF function will differ drastically because the collocation technique requires too many terms as compared with the KL expansion method. So the collocation technique predicted PDF would be Gaussian in almost all cases. Still, the collocation technique method is good for predicting the standard deviation or the autocovariance of the response, but not for predicting PDF function description. The proposed algorithms require deterministic responses to obtain stochastic response of the dynamic system when excited by a random process. For general application, these algorithms will be next applied to a continuous system using commercial finite element analysis software.

Appendix: Nonstationary Response of the SDOF System Subjected to an Exponential Autocovariance

The variance of SDOF system, $\sigma_x^2(t)$, when excited by stationary autocovariance with zero initial displacement and velocity is given as [5]

$$\begin{aligned}\sigma_x^2(t) = \int_{-\infty}^{\infty} \frac{S_{FF(\omega)}\,\mathrm{d}\omega}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\Bigg[1 + e^{-2\zeta\omega_n t}\Bigg\{1\\ + \frac{\omega_n}{\omega_d}\zeta\sin 2\omega_d t - e^{\zeta\omega_n t}\left(2\cos\omega_d t + \frac{2\omega_n}{\omega_d}\zeta\sin\omega_d t\right)\cos\omega t\\ - e^{\zeta\omega_n t}\frac{2\omega}{\omega_d}\zeta\sin\omega_d t\sin\omega t + \frac{(\zeta\omega_n)^2-\omega_d^2+\omega^2}{\omega_d^2}\sin^2\omega_d t\Bigg\}\Bigg]\end{aligned} \tag{A1}$$

Equation (A1) requires the definition of power spectral density function of the autocovariance. The PSD of exponential autocovariance function can be defined:

$$S_{FF}(\omega) = \frac{C_1}{\alpha^2+\omega^2} \tag{A2}$$

where C_1 and α are constants, which define exponential autocovariance distribution. The transient displacement variance of the SDOF system when excited by an exponential autocovariance is given as

$$\begin{aligned}\sigma_x^2(t) = \left[1 + e^{-2\zeta\omega_n t}\left(1 + \frac{\omega_n}{\omega_d}\zeta\sin(2\omega_d t)\right)\right]I_1\\ - 2\left[e^{-2\zeta\omega_n t}\left(\cos\omega_d t + \frac{\omega_n}{\omega_d}\zeta\sin\omega_d t\right)\right]I_2\\ - 2\left[\frac{e^{-2\zeta\omega_n t}}{\omega_d}\sin\omega_d t\right]I_3 + \frac{e^{-2\zeta\omega_n t}}{\omega_d^2}\sin^2\omega_d t[((\zeta\omega_n)^2\\ - \omega_d^2)I_1 + I_4]\end{aligned} \tag{A3}$$

where I_1, I_2, A, I_3, and I_4 are given as

$$I_1 = \frac{\pi(\alpha+2\zeta\omega_n)C_1}{2\zeta\omega_n^3\alpha(\alpha^2+\omega_n^2+2\zeta\omega_n\alpha)}$$

$$\begin{aligned}I_2 = \pi A C_1\Bigg[\frac{e^{-\alpha t}}{\alpha} - \frac{e^{-\zeta\omega_n t}}{\zeta\sqrt{1-\zeta^2}}(\sqrt{1-\zeta^2}\cos\omega_d t - \zeta\sin\omega_d t)\\ + \frac{(\alpha^2+2\omega_n^2(1-2\zeta^2))(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t)}{2\omega_n^3\zeta\sqrt{1-\zeta^2}}\Bigg]\end{aligned}$$

$$A = \frac{1}{(\alpha^2+\omega_n^2)^2-(2\alpha\zeta\omega_n)^2}$$

$$\begin{aligned}I_3 = \pi A C_1\Bigg[e^{-\alpha t} - \frac{e^{-\zeta\omega_n t}}{2\zeta\sqrt{1-\zeta^2}}\Bigg((2\zeta\sqrt{1-\zeta^2})\cos\omega_d t\\ - \left(2\zeta^2 - 1 + \frac{\alpha^2+2\omega_n^2(1-2\zeta^2)}{\omega_n^2}\right)\sin\omega_d t\Bigg)\Bigg]\end{aligned}$$

$$I_4 = \frac{\pi C_1}{2\zeta\omega_n(\alpha^2+2\alpha\zeta\omega_n+\omega_n^2)} \tag{A4}$$

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